A Study on Updating Spins in Ising Model to Solve Combinatorial Optimization Problems

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Abstract— Ising model, which consists of spins and interactions of them, is a novel way to solve combinatorial optimization problems, for example, LSI layout problem. The problem is solved by updating the spins stochastically after being mapped to the model. Spins can be updated simultaneously on hardware. However, the problems aren't solved fast since two spins with interaction should not be updated simultaneously. In this paper, we give a guideline of updating the spins simultaneously to execute a high-speed search and confirm it through experiments.

I. Introduction

A combinatorial optimization problem is to find an optimal discrete combination which minimizes (maximizes) an objective function. When the problem is NP-hard, as the size of this problem becomes big, the time which is needed to find the optimal solution increases rapidly. So, methods of stochastic search, for example, simulated annealing (SA) and genetic algorithm (GA), are proposed for finding a suboptimal solution with practical time.

The Ising computer is an alternative method of finding a suboptimal solution. In this computer, an Ising model, which models the behavior of the magnetic spins, is used to solve the problems. Every spin in the Ising model can be upward or downward. There are interactions between spins. To solve a combinatorial optimization problem using an Ising model, it is mapped to the Ising model, and spins are updated stochastically so that the energy of the Ising model is minimum. Since an Ising model is easy to be embedded in hardware and updating spins in parallel can be executed easily, the Ising computers are expected to execute a high-speed search for a solution of equivalent evaluation to a solution by existing methods. Various Ising computers were proposed[1, 2, 3, 4, 5]. In addition, some problems, e.g., a packing problem, which is used for LSI floorplan problem, were mapped to the Ising model and solved by these computers[6, 7, 8].

In the Ising computers, spins without interactions can be updated simultaneously[9]. In contrast, if spins with interactions are allowed to be updated simultaneously, the spin state may oscillate. Even if the oscillation does not occur, a solution obtained may be inferior to one by not updating spins with interactions simultaneously. Therefore, we consider that any pair of spins with an interaction should not be updated simultaneously. However, in case that almost all pairs of spins have the interactions, almost no pair of spins can be updated simultaneously. Then, searching for the solution cannot be executed with high speed. In order to execute a high-speed search, we desire to update some pairs of spins with interactions simultaneously.

In this paper, we will give a guideline so that as many pairs of spins with interactions as possible can be updated simultaneously under the condition to search for a solution of good evaluation. Then, in accordance with the guideline, we will propose a method of updating pairs of spins simultaneously for a traveling salesman problem. In addition, we will compare the method with other methods through computer experiments.

The rest of this paper is organized as follows. II explains an overview of an Ising model and Ising computers. III describes existing researches about mapping and updating spins stochastically. In IV, we discuss a motivation of our research and introduce a guideline for updating spins with interactions. In V, we propose an updating method with the guideline and make experimental comparisons. Finally, in VI, we conclude.

II. Ising Computer

A. Ising Model[7]

An Ising model expresses the behavior of the magnetic spins. Fig.1 shows an Ising model. The model consists of spins, which can be upward or downward, interactions between pairs of spins, and an external magnetic field.

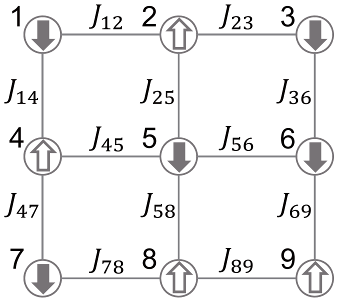


Fig. 1. An example of 2-D lattice Ising model.

The Ising model is represented by an undirected graph $G(V, E)$, where V is a set of vertices corresponding to spins and E is a set of weighted edges corresponding to interactions. Vertex i corresponds to spin i, and edge (i, j) corresponds to an interaction between spin i and j. The edge has weight J_{ij} which is an interaction coefficient. In this paper, when two spins have an interaction, they are called "connected".

In this model, a spin state changes to minimize the entire energy:

$$H = -\sum_i \sum_j J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i, \tag{1}$$

where σ_i denotes a value of spin i which takes 1 if the spin is upward and -1 if the spin is downward, and h_i is a coefficient of the external magnetic field. All spin values are updated to minimize the energy H. In particular, a spin state with minimum energy is called ground-state.

B. *Ising Computers*

To solve a combinatorial optimization problem using an Ising computer, at first, we map the problem to an Ising model on the Ising computer so that ground-state corresponds to an optimal solution of the problem. Next, the spins are updated in order to minimize the energy of the Ising model. Updating a spin means that whether to accept a spin flip is determined. Finally, a solution of the problem is gotten from a spin state obtained. Since an Ising model is easy to be embedded in hardware and parallel operations can be executed easily, the Ising computers are expected to execute high-speed-search for a suboptimal solution.

III. Searching Solutions Using Ising Model

To solve a combinatorial optimization problem using an Ising model, the problem is mapped to the Ising model such that the ground-state of it corresponds to the optimal solution of the problem. Then, searching for the ground-state is executed by updating spins stochastically. In this section, we explain a method to map TSP (Traveling Salesman Problem) to the Ising model and the flow of searching solutions.

A. *Mapping TSP to the Ising Model[2, 10]*

TSP is defined as follows: given a set of cities and the distance for each pair of cities, find the shortest cycle to visit all cities exactly once. Note that the traveler returns directly to the starting city from the last city. To map TSP to the Ising model, we define the binary variable $x_{i,a}$ which is 1 if city i is visited at order a, and 0 otherwise. Note that after the mapping, not $x_{i,a}$ but the spin values are updated to search solutions. In this paper, we explain the mapping by using $x_{i,a}$, because it is convenient to describe the energies in terms of $x_{i,a}$. The binary variable $x_{i,a} = 1$ is converted to the spin value $\sigma_{i,a} = 1$, and $x_{i,a} = 0$ is converted to $\sigma_{i,a} = -1$ by

$$x_{i,a} = (\sigma_{i,a} + 1)/2. \tag{2}$$

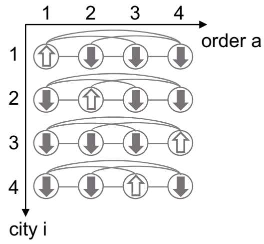


Fig. 2. Edges mean the interactions to enforce that every city appears exactly once in a cycle.

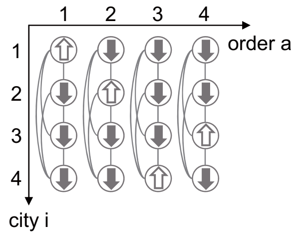


Fig. 3. Edges mean the interactions to enforce that exactly one city is visited at each order a.

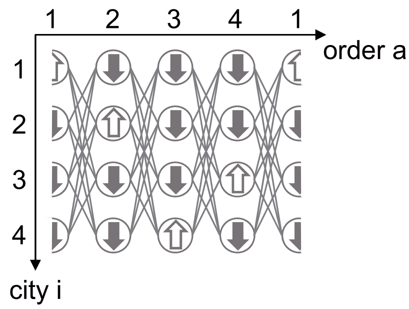


Fig. 4. Edges mean the interactions which make H_B equal to the total distance of the cycle which is decided by a set of $x_{i,a}$. The interaction between two spins corresponds to the distance between two cities.

The energy of the Ising model which TSP is mapped to is $H = wH_A + H_B$; wH_A is a penalty term and H_B is the total distance of the cycle which is decided by a set of $x_{i,a}$. A parameter w is a weight of the penalty term. H_A is defined by

$$H_A = \sum_{i=1}^{N}\left(\sum_{a=1}^{N} x_{i,a} - 1\right)^2 + \sum_{a=1}^{N}\left(\sum_{i=1}^{N} x_{i,a} - 1\right)^2, \tag{3}$$

which represents two constraints: the first term enforces that every city can appear exactly once in a cycle and the second term enforces that exactly one city is visited at each order a. As shown in Fig.2 and Fig.3, the interactions of H_A correspond to edges connecting spins. In the following, these edges are called "*penalty edges*". These interactions enforce that spin values should not violate the constraints. If $H_A = 0$, the solution represented by the spin state is feasible.

H_B, the total distance of the cycle which is decided by a set of $x_{i,a}$, is defined by

$$H_B = \sum_{a=1}^{N}\sum_{i=1}^{N}\sum_{i'=1}^{N} d_{i,i'} x_{i,a} x_{i',a+1}, \tag{4}$$

where $x_{i,N+1} = x_{i,1}$, and $d_{i,i'}$ is a distance between city i and city i'. The interactions in Fig.4 make H_B equal to the total distance of the cycle which is decided by a set of $x_{i,a}$. In the following, the edges in Fig.4 are called "*distance edges*".

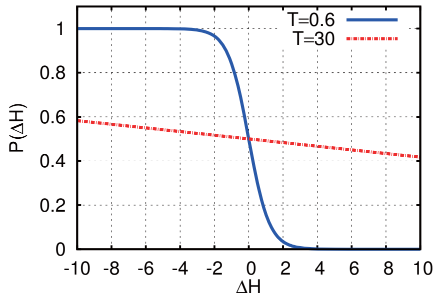


Fig. 5. The probability $P(\Delta H)$ for $T = 0.6$ and $T = 30$

The parameter w, the weight of the penalty term, has enough weight so that evaluations of feasible solutions better than that of infeasible solutions.

B. Updating Spins with Some Probability[3]

In this paper, we use the search method based on Hitachi's Ising computer implemented on FPGA[3], because an evaluation of the solution is reported to be equivalent to a solution by simulated annealing.

The flow of searching solutions by updating spins in the Ising model is as follows:

Step1. Temperature T is set to a given initial value. All spin values are set to 1 or -1 randomly.

Step2. Repeat the following step until T reaches a given final value.

Step2-1. Every spin is updated stochastically. T is decreased as $T = \beta T$, where β is a cooling ratio $(0 < \beta < 1)$.

In Step2-1, a spin flip is accepted with the following probability

$$P(\Delta H) = \frac{1}{2}\left\{1 + \tanh\left(-\frac{\Delta H}{2T}\right)\right\}, \tag{5}$$

where ΔH is the increase of the energy by spin flip. Fig.5 shows this probability. In case that T is high, the global search for the spin state can be executed because a spin flip which increases the energy is accepted with probability which is nearly equal to the probability of accepting a spin flip which decreases the energy. On the other hand, at a low temperature, local search can be executed because a spin flip which increases the energy is hard to be accepted and a spin flip which decreases the energy is accepted with high probability.

IV. Motivation of This Research and Proposed Guideline

In Ising computers, spins which are not connected can be updated simultaneously[9]. For example, when an Ising model is represented by a grid graph[11], all spins can be grouped into two groups, each of which doesn't has connected spins because the grid graph is bipartite. Since all spins in one group can be updated simultaneously, the number of times which is necessary for updating all spins in the Ising model is smaller than the number of times to update spins one by one.

On the other hand, if connected spins are allowed to be updated simultaneously, the spin state may oscillate. Even if the spin state does not oscillate, the solution is not always inferior to that by updating spins one by one. Thus, connected spins should not be updated simultaneously.

However, almost no pair of spins can be updated simultaneously if the Ising model can be represented by a dense graph: the number of times which is necessary for updating all spins is nearly equal to the number of spins in the model. Therefore, searching solutions on hardware cannot perform with high-speed. In order to execute a high-speed search, we desire to update some pairs of connected spins simultaneously.

If spins connected by an edge are updated simultaneously, energy increase related to weight of the edge may occur though each spin is updated to decrease the energy. Therefore, a guideline for updating connected spins simultaneously is as follows:

Guideline: As many spins as possible are updated simultaneously except for connected spins by edges of large weights.

In the following, we will propose a method to update spins simultaneously in accordance with the guideline.

V. Computer Experiments

We experiment to confirm the above guideline. Recall that the problem which is mapped to an Ising model is TSP. Let the number of cities be N: i.e., the number of all spins is N^2.

A. Grouping Spins into Subsets

To explain a method of updating connected spins simultaneously under the proposed guideline, we will group all spins in the Ising model into subsets. Each subset is focused on one by one and all spins in it are updated simultaneously. Updating all spins in one subset simultaneously is called "subset-updating" in this paper. The number of subset-updatings necessary for updating all spins is equal to the number of subsets.

We map TSP to an Ising model, so the guideline which TSP is adapted to is as follows: as many connected spins as possible are updated simultaneously except updating spins connected by penalty edges. Note that weight of a penalty edge is large enough to avoid an infeasible solution. In order to experiment with the guideline for TSP, we explain three kinds of groupings; (1) two spins connected are not updated simultaneously, (2) two spins connected by distance edges and penalty edges are allowed to be updated simultaneously, and (3) two spins connected by distance edges are allowed to be updated simultaneously while that connected by penalty edges are not allowed to be updated simultaneously.

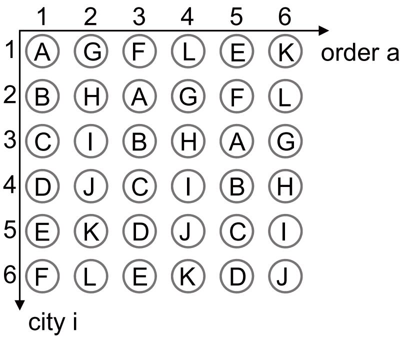


Fig. 6. All spins in an Ising model are grouped into $2N$ subsets $A, B, \cdots$, and L by "*$2N$-partite Grouping*".

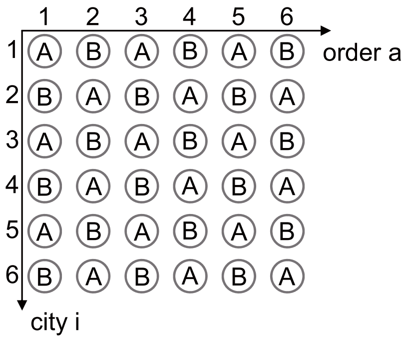


Fig. 7. All spins in an Ising model are grouped into two subsets A, B by "*Grouping Like Checkerboard*".

G-1. Grouping Singly

By "*Grouping Singly*", connected spins are not updated simultaneously. All spins are grouped into separate subsets; i.e., each subset has one spin. Therefore, all spins are updated one by one.

Since the number of subsets is N^2, the number of subset-updatings necessary for updating all spins is N^2.

We expect that subset-updatings under this grouping can search for a suboptimal solution, so the energies of the solutions under the other grouping will be compared with that of *Grouping Singly*.

G-2. $2N$-partite Grouping

By "*$2N$-partite Grouping*", connected spins are not updated simultaneously. All spins are grouped into subsets such that the number of subsets is minimum under the constraint which each subset doesn't have connected spins. A spin of $\sigma_{i,a}$ ($i = 1, 2, \cdots, N$ and $a = 1, 2$) and a spin $\sigma_{i',a'}$ are included in one subset, where i' is $((i + k - 1) \bmod N) + 1$ and a' is $a + 2k$ ($1 \leq k < N/2$). As shown in Fig.6, connected spins are not included in each subset.

This grouping generates $2N$ subsets; i.e., the number of subset-updatings necessary for updating all spins is $2N$. Therefore, searching solutions by subset-updatings under this grouping may be executed $N/2$ times faster than that of **G-1**.

By this grouping, since connected spins are not updated simultaneously, we expect that the energies of solutions under this grouping are nearly equal to that of **G-1**.

G-3. One Grouping

Unlike the above groupings, by "*One Grouping*", connected spins may be updated simultaneously. All spins are put into one subset; they may be updated simultaneously.

Since there is only one subset, all spins may be updated in constant time, regardless of the number of spins. Therefore, searching solutions under this grouping may be executed N^2 times faster than that of **G-1**.

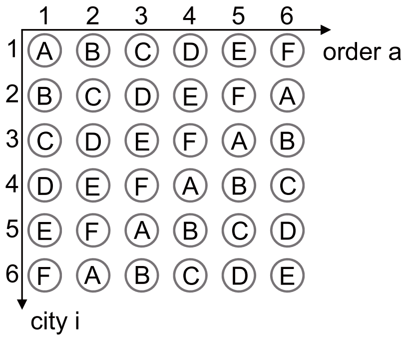


Fig. 8. All spins in an Ising model are grouped into N subsets $A, B, \cdots$, and F by "*Moderate Grouping*".

G-4. Grouping Like Checkerboard

By "*Grouping Like Checkerboard*", connected spins may be updated simultaneously. All spins are grouped into two subsets like a checkerboard pattern. Fig.7 shows this grouping.

Since there are two subsets by this grouping, all spins may be updated in constant time, regardless of the number of spins. Therefore, searching solutions under this grouping may be executed $N^2/2$ times faster than that of **G-1**.

G-5. Moderate Grouping

"*Moderate Grouping*" is a proposed grouping in accordance with the guideline. Spins connected by penalty edges are not updated simultaneously while spins connected by distance edges may be updated simultaneously. A spin of $\sigma_{i,1}$ ($i = 1, 2, \cdots, N$) and a spin of $\sigma_{i',a'}$ are in one subset, where i' is $(N - (k + 1) + i) \bmod N + 1$ and a' is $1 + k$ ($1 \leq k < N$). As shown in Fig.8, spins connected by penalty edges are not included in one subset, but spins connected by distance edges may be included in one subset.

The number of subsets in this grouping is N; i.e., the number of subset-updatings necessary for updating all spins is N. Therefore, searching solutions under this grouping may be executed N times faster than that of

G-1.

Comparison of the above groupings is shown in the left five columns of Table.I.

B. Experimental Results

We used TSP problem “fri26” in TSPLIB[12] for the experiments.The number of cities in this problem is 26 and the total distance of the shortest cycle is 937.

As explained in III, since weight of the penalty edge should be larger than a distance between two cities in order to search for a feasible solution. Therefore, the weight coefficient w of the penalty term was set to 150. Initial temperature and final temperature were determined by pre-runs. The cooling ratio β was calculated such that the number of iterations in Step2 is $10 \cdot 2^1, 10 \cdot 2^2, \cdots, 10 \cdot 2^{24}$. We expect that there is a trade-off between the solution evaluation and the number of iterations in Step2, similar to SA. Therefore, we experimented for five times with distinct seeds of pseudorandom numbers for each cooling ratio.

G-1, G-2, G-5 Fig.9(a), 9(b), 9(c) show ratio of resultant energy to the total distance of the shortest cycle in fri26. Note that the total distance is an energy of a solution and so the resultant ratio is 1.0 when the solution is optimal. As shown in these figures, there is the above trade-off. In addition, the number of infeasible solutions is large when the number of iterations in step2 is small, while the number of infeasible solutions is zero when the number of iterations in Step2 is large.

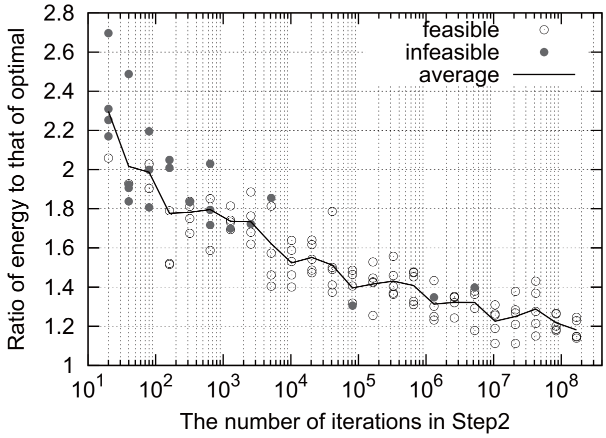
Fig.10 shows the comparison of the averages of the ratios in these three groupings. As shown in this figure, the energy of the proposed method **G-5** is nearly equal to that of the other two groupings.

G-3 All solutions under this grouping are infeasible, where spin values are all 1 or all -1.

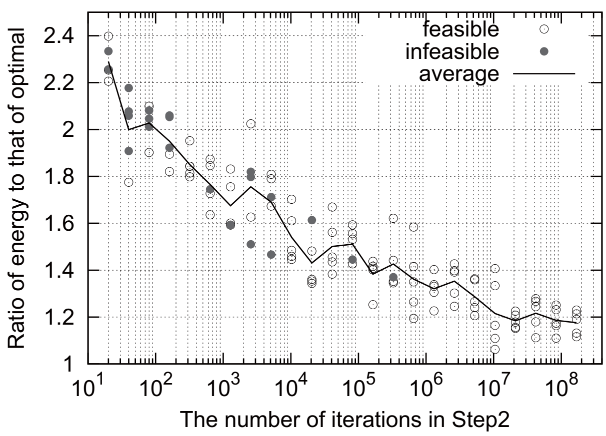
G-4 All solutions under this grouping are also infeasible. The transition of spin state is occurred by repetition of subset-updatings as shown in Fig.11.

Comparison of property and experimental results of five kinds of groupings are shown in Table.I. As shown in this table, connected spins may be updated simultaneously under **G-3**, **G-4**, and **G-5**. Feasible solutions are obtained under **G-5**, on the other hand, solutions under **G-3**, **G-4** are infeasible. Since **G-5** differs from **G-3**, **G-4** in not updating spins connected by penalty edges simultaneously, we consider that if spins connected by the penalty edges are not updated simultaneously, a feasible solution can be obtained without oscillation of spin state.

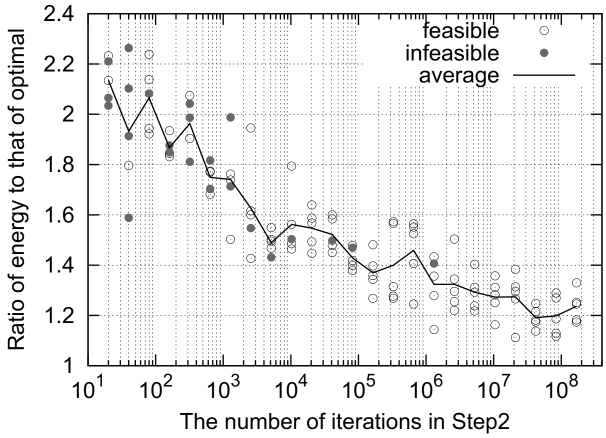
Recall that the number of subset-updatings necessary for updating all spins is equal to the number of subsets; i.e., the necessary time to search for a solution must be proportional to the number of subsets when to search solutions is executed on hardware. Therefore, we expect that searching for a suboptimal solution under **G-5** is more efficient than that under **G-1**.



(a) Resultant ratio under **G-1**



(b) Resultant ratio under **G-2**



(c) Resultant ratio under **G-5**

Fig. 9. The ratio of resultant energy to that of the optimal solution. “feasible” and “infeasible” mean the ratios of the energies of feasible (infeasible) solutions. “average” means an average of the ratios of feasible solutions.

VI. Conclusion

In searching solutions using an Ising model, connected spins should not be updated simultaneously, but a high-speed search cannot be executed when most of spins are connected. In order to execute the search, we gave a

TABLE I
Comparison of five groupings. "Ratio of solution energy" is the ratio of the resultant energy to the optimal energy, where a solution energy was obtained when the number of iterations in Step2 is $10 \cdot 2^{24}$.

Grouping	In a subset, spins connected by		the number of subsets	the number of spins in a subset	ratio of solution energy
	distance edges	penalty edges			
G-1: Singly	not included	not included	N^2	1	1.18
G-2: $2N$-partite	not included	not included	$2N$	$N/2$	1.18
G-5: Moderate	included	not included	N	N	1.24
G-4: Checkerboard	included	included	2	$N^2/2$	infeasible
G-3: One	included	included	1	N^2	infeasible

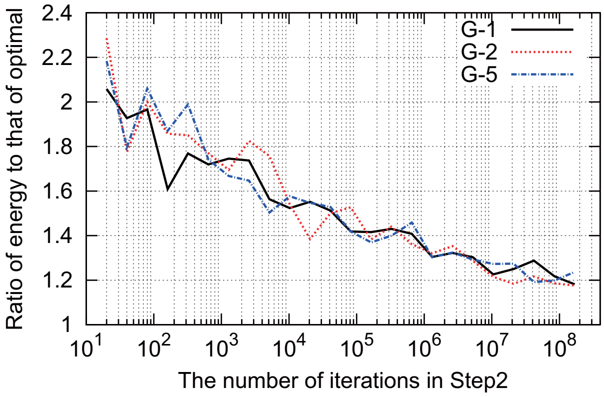


Fig. 10. Comparison of averages of the ratio under **G-1**,**G-2** and **G-5**

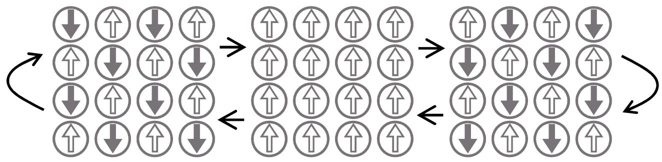


Fig. 11. Transition of spin state by a repetition of subset-updatings under **G-4**

guideline that as many spins as possible are updated simultaneously except for connected spins by edges of large weights. In addition, we proposed a Moderate Grouping, to update spins efficiently in accordance with the guideline. The grouping is compared with the other groupings by experiments. As a result, the evaluation of solutions under the grouping is nearly equal to that under the best grouping.

Future works include thinking of a groping method in accordance with the guideline for a problem where interactions have many different weights.

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